Phase flows and vectorial lagrangians in $J^3(\pi)$

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Abstract. On the basis of Liouville theorem the generalization of the Nambu mechanics is considered. For three-dimensional phase space the concept of vector hamiltonian and vector lagrangian is entered.

1. Standard phenomenological approach to the analysis dynamic system is the construction for it the functional of actions $S = \int L\, dt$. We represent this functional as submanifolds in jet bundles $J^n(\pi)$: $E \to M$

$$F(t, x_0, x_1, ..., x_n) = 0,$$

where $t \in M \subset R$, $u = x_0 \in U \subset R$, $x_i \in J^i(\pi) \subset R^n$, $E = M \times U$.

The Euler-Lagrange equation

$$\sum_{k=0}^{n} (-)^k \frac{d^k}{dt^k} \frac{\partial L}{\partial x_k} = 0 \tag{1}$$

describes a lines (jet) in $J^{2n}(\pi)$. Embedding $J^n(\pi) \subset J^1(J^1(J^1(...J^1(\pi))))$ allows us to rewrite differential equation n-st order as system of the n equations 1-st order

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}. \tag{2}$$

According to the Noether theorem, symmetry of functional S with respect to generator $X = \partial/\partial t$ give us the conservation law I, and hamiltonian form for our dynamics:

$$\dot{\mathbf{x}} = \{H(I), \mathbf{x}\}. \tag{3}$$

2. Another approach for receiving of the Euler-Lagrange equation (1) for every hamiltonians set was described by P.A.Griffiths [1]. He find such 1-form

$$\psi = Ldt + \lambda^i \theta_i, \qquad i = 0..n-1,$$

which does not vary at pullback along vector fields $(\partial/\partial\theta_i, \partial/\partial d\theta_{n-1})$. Here

$$\theta_i = dx_i - x_{i+1}\, dt$$

is the contact distribution, λ^i is the Lagrange multipliers. Bounding of the form $\Psi = d\psi$ on the field $(\partial/\partial\theta_i, \partial/\partial d\theta_{n-1})$ gives the set

$$\begin{cases} \dfrac{\partial L}{\partial x_i} = \dfrac{d\lambda^i}{dt} + \lambda^{i-1}, \qquad i = 0..n-1, \\[2ex] \dfrac{\partial L}{\partial x_n} = \lambda^{n-1}, \end{cases}$$

which is equivalent to the equation (1).

Hamiltonian formulation of this theory assume that Lagrange's multipliers λ^i be a dynamic variables $H = H(x_i, \lambda^i)$:

$$\psi = (L - \lambda^i x_{i+1}) \wedge dt + \lambda^i dx_i = -H dt + \lambda^i dx_i.$$

Then bounding of the form $\Psi = d\psi$ on the field $(\partial/\partial x_i, \partial/\partial \lambda^i)$ gives

$$\frac{\partial H}{\partial x_i} = -\frac{\partial \lambda^i}{\partial t}, \qquad \frac{\partial H}{\partial \lambda^i} = \frac{\partial x_i}{\partial t}.$$

3. Our target is the generalization of the above scheme on a case odd jets. To clear idea of a method we shall receive the Euler-Lagrange equation for $L \in J^3(\pi)$.

Theorem 1. For $L \in J^3(\pi)$ the Euler-Lagrange equation has the form

$$\frac{1}{2}\frac{d}{dt}\left(L^i_{\dot{x}_k} - L^k_{\dot{x}_i}\right) = L^k_{x_i} - L^i_{x_k}. \tag{4}$$

Proof. Let ψ be the Griffiths 2-form:

$$\psi = L^i dx_i \wedge dt + \lambda^i \Theta_i,$$

where $\Theta = \theta \wedge \theta$. Exterior differencial this form is

$$\begin{aligned}
d\psi &= dL^i \wedge \omega_i = (\operatorname{rot} L)^k\, \Theta_k \wedge dt + d\lambda^i \wedge \Theta_i + \lambda^i \wedge d\Theta_i + \\
&+ L^2_{\dot{x}_3}\, d\dot{x}_3 \wedge dx_2 \wedge dt + L^3_{\dot{x}_2}\, d\dot{x}_2 \wedge dx_3 \wedge dt \\
&+ L^3_{\dot{x}_1}\, d\dot{x}_1 \wedge dx_3 \wedge dt + L^1_{\dot{x}_3}\, d\dot{x}_3 \wedge dx_1 \wedge dt \\
&+ L^1_{\dot{x}_2}\, d\dot{x}_2 \wedge dx_1 \wedge dt + L^2_{\dot{x}_1}\, d\dot{x}_1 \wedge dx_2 \wedge dt \\
&+ L^1_{\dot{x}_1}\, d\dot{x}_1 \wedge dx_1 \wedge dt + L^2_{\dot{x}_2}\, d\dot{x}_2 \wedge dx_2 \wedge dt + L^3_{\dot{x}_3}\, d\dot{x}_3 \wedge dx_3 \wedge dt.
\end{aligned}$$

Limiting it on vector fields $v = (\partial_{\Theta_k}, \partial_{d\Theta_k})$,

$$(\operatorname{rot} L)^k = -\dot{\lambda}^k, \tag{5}$$

$$L^3_{\dot{x}_2} - L^2_{\dot{x}_3} = 2\lambda^1, \qquad L^1_{\dot{x}_3} - L^3_{\dot{x}_1} = 2\lambda^2, \qquad L^2_{\dot{x}_1} - L^1_{\dot{x}_2} = 2\lambda^3$$

we get the Euler-Lagrange equation (4).

4. Now we consider construction of the vector hamiltonian h^i for $L \in J^3(\pi)$. Rewrite the Griffiths 2-form ψ as

$$\begin{aligned}
\psi &= L^i \omega_i + \lambda^i \Theta_i \\
&= \left(L^1 - \left(\lambda^3 \dot{x}_2 - \lambda^2 \dot{x}_3\right)\right) dx_1 \wedge dt \\
&+ \left(L^2 - \left(\lambda^1 \dot{x}_3 - \lambda^3 \dot{x}_1\right)\right) dx_2 \wedge dt \\
&+ \left(L^3 - \left(\lambda^2 \dot{x}_1 - \lambda^1 \dot{x}_2\right)\right) dx_3 \wedge dt + \lambda^i dS_i \\
&= -h^i dx_i \wedge dt + \lambda^i dS_i.
\end{aligned}$$

Here $dS_i = \varepsilon_{ijk} dx_j \wedge dx_k$ be a Plücker coordinats of area element dS spanned by vectors dx_i.

Definition 1. The vector field $\mathbf{f}$ is called conservative if

$$\operatorname{div} \mathbf{f} = 0.$$

In other words, conservative vector field is divergence-free.

Definition 2. Phase trajectory $\mathbf{x}(t)$ is called Lagrange-stable if for all $t > 0$ remains in some bounded domain of phase space. Geometrically it means, that a phase flow (2) should be divergence-free.

Theorem 2. The Lagrange-stable phase flow is hamiltonians.

Proof. We first calculate the exterior derivatives of closed 2-forms ψ:

$$d\psi = (\operatorname{rot} \mathbf{h})^k \, \Theta_k \wedge dt + \dot{\lambda}^k dt \wedge dS_k + \operatorname{div} \lambda^k \cdot dV = 0.$$

Then from $\operatorname{div} \lambda^k = 0$ it follows that

$$\dot{\lambda} = \operatorname{rot} \mathbf{h}.$$

I.e. from hamiltonians point of view the set (2) described of dynamics of a generaliszed moments λ, which were defined in (5).

5. The base of deformation quantization of dynamical system in $J^2(\pi)$ is the Liouville theorem about preserved of the volume $\Omega = dx_0 \wedge dx_1$ by phase flows. Geometrically it means, that Lie derivative of the 2-form Ω along vector field X_H^1 is zero: $\mathcal{L}_X \Omega = 0$. In other words if $\{g_t\}$ denotes the one parameter group symplectic diffeomorphisms generated by vector fields X_H^1, then $g_t^* \Omega = \Omega$ and the phase flow $\{g_t\}$ preserved the volume form Ω.

For extended this construction on $J^3(\pi)$ we consider the 3-form of the phase space volume

$$\Omega = dx_0 \wedge dx_1 \wedge dx_2.$$

Theorem 3. The volume 3-form $\Omega \in \Lambda^3$ supposes existence two polyvector hamiltonians fields $X_H^1 \in \Lambda^1$ and $X_H^2 \in \Lambda^2$.

Proof. By definition, put

$$\mathcal{L}_X \Omega = X \rfloor d\Omega + d\,(X \rfloor \Omega) = 0.$$

Since $\Omega \in \Lambda^3$, we see that $d\Omega = 0$ and

$$d\,(X \rfloor \Omega) = 0.$$

From Poincare's lemma it follows that form $X \rfloor \Omega$ is exact, and

$$X \rfloor \Omega = \Theta = d\mathbf{H}.$$

1) If $\mathbf{X}_H^1 \in \Lambda^1$, then $\Theta \in \Lambda^2$, $\mathbf{H} = (\mathbf{h} \cdot d\mathbf{x}) \in \Lambda^1$. Hamiltonian vector fields has the form

$$\begin{aligned} X_H^1 &= (\operatorname{rot}\mathbf{h} \cdot \frac{\partial}{\partial \mathbf{x}}) \\ &= \left(\frac{\partial h_2}{\partial x_1} - \frac{\partial h_1}{\partial x_2}\right)\frac{\partial}{\partial x_0} + \left(\frac{\partial h_0}{\partial x_2} - \frac{\partial h_2}{\partial x_0}\right)\frac{\partial}{\partial x_1} + \left(\frac{\partial h_1}{\partial x_0} - \frac{\partial h_0}{\partial x_1}\right)\frac{\partial}{\partial x_2}. \end{aligned} \tag{6}$$

2) If $X_H^2 \in \Lambda^2$, then $\Theta \in \Lambda^1$, $H \in \Lambda^0$ and we see already hamiltonian bivector fields

$$X_H^2 = \frac{1}{2}\left(\frac{\partial H}{\partial x_0}\cdot\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2} + \frac{\partial H}{\partial x_1}\cdot\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_0} + \frac{\partial H}{\partial x_2}\cdot\frac{\partial}{\partial x_0}\wedge\frac{\partial}{\partial x_1}\right). \tag{7}$$

More generalized (but scalar) construction was considered in [2].

Poisson's bracket for vectorial hamiltonian (6) has the form

$$\begin{aligned} \{\mathbf{h}, G\} &= X_H^1 \rfloor dG \\ &= \left(\frac{\partial h_2}{\partial x_1} - \frac{\partial h_1}{\partial x_2}\right)\frac{\partial G}{\partial x_0} + \left(\frac{\partial h_0}{\partial x_2} - \frac{\partial h_2}{\partial x_0}\right)\frac{\partial G}{\partial x_1} + \left(\frac{\partial h_1}{\partial x_0} - \frac{\partial h_0}{\partial x_1}\right)\frac{\partial G}{\partial x_2}, \end{aligned}$$

and dynamic equations is (2)

$$\dot{\mathbf{x}} = \{\mathbf{h}, \mathbf{x}\}. \tag{8}$$

Poisson's bracket for bivector fields requires introduction two hamiltonians

$$\begin{aligned} X_H^2\rfloor(dF\wedge dG) = \{H, F, G\} = \frac{1}{2}\Bigg[\frac{\partial H}{\partial x_0}\cdot\left(\frac{\partial F}{\partial x_1}\frac{\partial G}{\partial x_2} - \frac{\partial F}{\partial x_2}\frac{\partial G}{\partial x_1}\right) \\ + \frac{\partial H}{\partial x_1}\cdot\left(\frac{\partial F}{\partial x_2}\frac{\partial G}{\partial x_0} - \frac{\partial F}{\partial x_0}\frac{\partial G}{\partial x_2}\right) + \frac{\partial H}{\partial x_2}\cdot\left(\frac{\partial F}{\partial x_0}\frac{\partial G}{\partial x_1} - \frac{\partial F}{\partial x_1}\frac{\partial G}{\partial x_0}\right)\Bigg], \end{aligned} \tag{9}$$

such that dynamic equations (2) has the Nambu form [3]

$$\dot{\mathbf{x}} = \{F, G, \mathbf{x}\}.$$

Example. Consider the dynamics of Frenet frame with constant curvature and torsion

$$\begin{cases} \dot{x} = y \\ \dot{y} = z - x \\ \dot{z} = -y \end{cases} \tag{10}$$

Lax representations for this set has the form

$$\dot{A} = [A, B], \quad A = \begin{pmatrix} x & y & x \\ y & 2z & y \\ x & y & x \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

and gives us following invariants

$$I_k = \frac{1}{k}\mathrm{Tr}\mathbf{A}^k,$$

$$I_1 = x + z, \quad I_2 = \frac{1}{2}(x^2 + y^2 + z^2), \quad I_3 = \frac{1}{3}\left(x^3 + \frac{3}{2}y^2(x+z) + z^3\right) ...$$

Let $H_1 = x + z$ and $H_2 = \frac{1}{2}(2xz - y^2)$ - are the hamiltonians of Frenet set, then

$$I_1 = H_1, \quad I_2 = \frac{1}{2}H_1^2 - H_2, \quad I_3 = \frac{1}{3}H_1\left(H_1^2 - 3H_2\right) ...$$

The system (2) is equivalent to system

$$\dot{\mathbf{x}} = \{H_1, H_2, \mathbf{x}\}$$

with a Poisson bracket (9).

For a finding of vectorial hamiltonian we write the differential $\Psi = d\psi$ of Lagrange's 1-form for Frenet set (10):

$$\Psi = ydy \wedge dz + (z - x)dz \wedge dx - ydx \wedge dy,$$

and, using homotopy formula, we get an expression for the vectorial hamiltonians h^i and vectorial lagranfians L^i:

$$\mathbf{h} = \frac{1}{3}\begin{pmatrix} y^2 + z^2 - xz \\ -y(x+z) \\ y^2 + x^2 - xz \end{pmatrix}, \qquad \mathbf{L} = \begin{pmatrix} z\dot{y} - y\dot{z} - h_1 \\ x\dot{z} - z\dot{x} - h_2 \\ y\dot{x} - x\dot{y} - h_3 \end{pmatrix}.$$

References

[1] P.A. Griffiths, *Exterior differential systems and the calculus of variations*, Birkhauser, Boston (1983).

[2] V.N. Dumachev, *Generalized Nambu dynamics and vectorial Hamiltonians*, arXiv: math.DG/0904.4326.

[3] Y. Nambu, Generalized Hamiltonian dynamics, *Phys.Rev.D*, **7** (1973), 5405-5412.